Computing Generator in Cyclotomic Integer Rings

A subfield algorithm for the Principal Ideal Problem in $L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$ and application to the cryptanalysis of a FHE scheme

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  1. Given the $\mathbb{Z}$-basis of the ideal $\mathfrak{a} = \langle \boldsymbol{g} \rangle$, find a — not necessarily short — generator $\boldsymbol{g'} = \boldsymbol{g} \cdot \boldsymbol{u}$ for a unit $\boldsymbol{u}$.
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Campbell, Groves, and Sheperd (2014) found a solution in polynomial time for the second point for power-of-two cyclotomic fields. Cramer, Ducas, Peikert, and Regev (2016) provided a proof and an extension to prime-power cyclotomic fields.

Key Generation:

1. Fix the security parameter $N = 2^n$.
2. Let $F(X) = X^N + 1$ be the polynomial defining the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta_{2N})$.
3. Set $G(X) = 1 + 2 \cdot S(X)$,
   for $S(X)$ of degree $N - 1$ with coefficients in $\left[-2^{\sqrt{N}}, 2^{\sqrt{N}}\right]$,
   such that the norm $\mathcal{N}\left(\langle G(\zeta_{2N})\rangle\right)$ is prime.
4. Set $\boldsymbol{g} = G(\zeta_{2N}) \in \mathcal{O}_{\mathbb{K}}$.
5. Return the secret key $\mathsf{sk} = \boldsymbol{g}$ and the public key $\mathsf{pk} = \mathsf{HNF}(\langle \boldsymbol{g} \rangle)$.

FHE scheme – Smart and Vercauteren PKC 2010

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Our goal: Recover the secret key from the public key.

Outline of the algorithm

1. Perform a reduction from the cyclotomic field to its totally real subfield, allowing to work in smaller dimension.
2. Then a $\mathfrak{q}$-descent makes the size of involved ideals decrease.
3. Collect relations and run linear algebra to construct small ideals and a generator.
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$$L_{|\Delta_{\mathbb{K}}|}(\alpha) = 2^{N^{\alpha+o(1)}}.$$

1. Reduction to the totally real subfield

Goal: Halving the dimension of the ambient field

Gentry-Szydlo algorithm: *Polynomial complexity*

- **Input:** a $\mathbb{Z}$-basis of $\mathcal{I} = \langle \boldsymbol{u} \rangle$ and $\boldsymbol{u} \cdot \bar{\boldsymbol{u}}$
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Problem: no information about $\boldsymbol{g} \cdot \bar{\boldsymbol{g}}$ ($\boldsymbol{g}$ is the private key)

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In the end, we get $\boldsymbol{g} \cdot \bar{\boldsymbol{g}}^{-1}$ and a $\mathbb{Z}$-basis of $\mathcal{I}^+ = \langle \boldsymbol{g} + \bar{\boldsymbol{g}} \rangle \subset \mathbb{Q}(\zeta + \zeta^{-1})$

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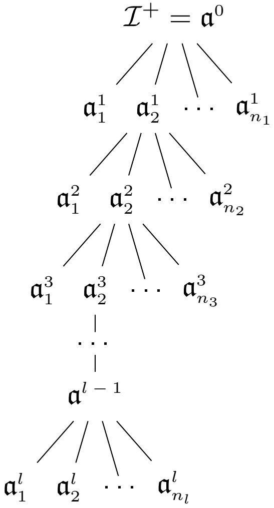
Once we have a generator for $\mathcal{I}^+$, we get one for $\mathcal{I}$ by multiplying by

$$\frac{1}{1 + \bar{\boldsymbol{g}} \cdot \boldsymbol{g}^{-1}}$$

2. The $\mathfrak{q}$-descent



Input ideal – Norm arbitrary large

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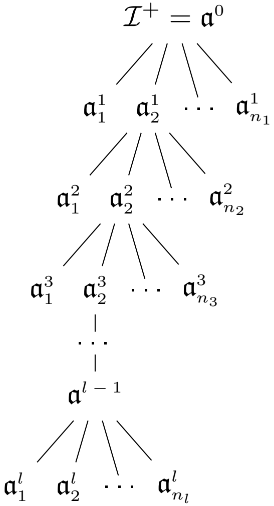
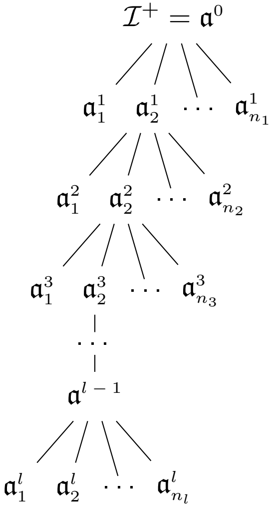


Input ideal – Norm arbitrary large

Initial reduction – Norm: $L_{|\Delta_{\mathbb{K}}|}\left(\frac{3}{2}\right)$



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Initial reduction – $L_{|\Delta_{\mathbb{K}}|}(1)$-smooth

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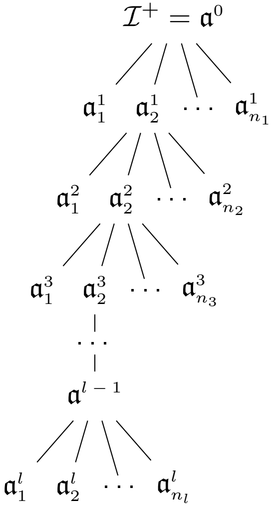

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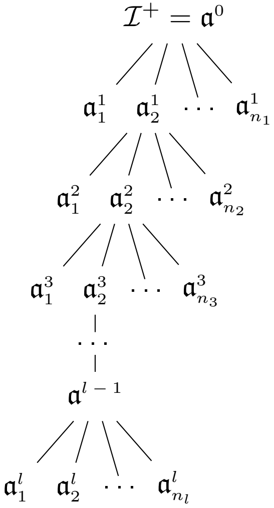
First step – Norm: $L_{|\Delta_{\mathbb{K}}|}\left(\frac{5}{4}\right)$



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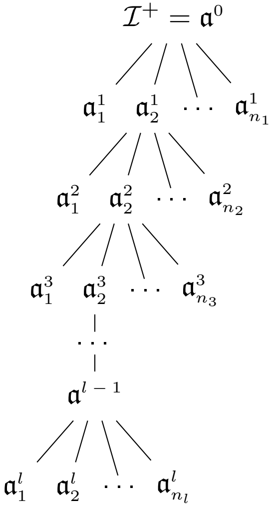

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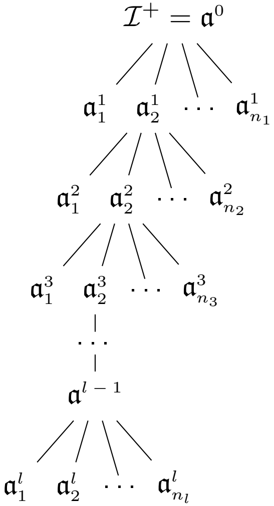
First step – $L_{|\Delta_{\mathbb{K}}|}\left(\frac{3}{4}\right)$-smooth

Second step – Norm: $L_{|\Delta_{\mathbb{K}}|}\left(\frac{9}{8}\right)$



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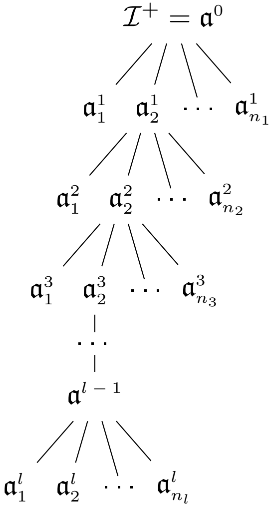

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Last but one step – Norm: $\approx L_{|\Delta_{\mathbb{K}}|}(1)$

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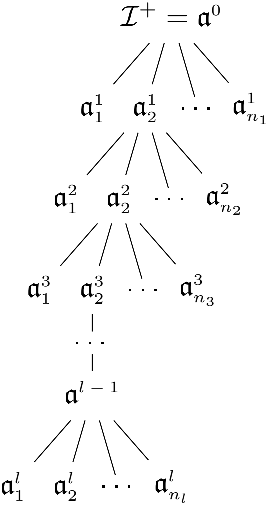

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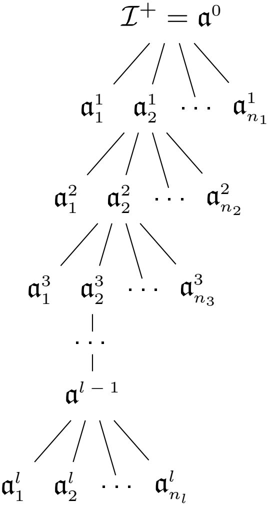

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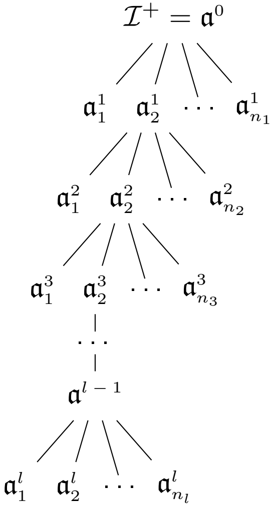
Last step – Norm: $L_{|\Delta_\mathbb{K}|}(1)$



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Output: small vector $\longleftrightarrow$ algebraic integer $\boldsymbol{v} \in \mathfrak{a}$

$\Longrightarrow$ ideal $\mathfrak{b} \subset \mathcal{O}_{\mathbb{K}^+}$ s.t. $\langle \boldsymbol{v} \rangle = \mathfrak{a} \cdot \mathfrak{b}$ and

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Cost: DBKZ-reduction $\Rightarrow$ Poly $(N, \log \mathcal{N}(\mathfrak{a})) \cdot L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$

Smoothness tests & Randomization

Heuristic

We assume that the probability $\mathcal{P}$ that an ideal of norm bounded by $L_{|\Delta_{\mathbb{K}}|}(a)$ is a power-product of prime ideals of norm bounded by $B = L_{|\Delta_{\mathbb{K}}|}(b)$ satisfies

$$\mathcal{P} \geq L_{|\Delta_{\mathbb{K}}|}\left(a-b\right)^{-1}.$$

Using ECM algorithm, each B-smoothness test costs $L_{|\Delta_{\mathbb{K}}|}\left(\frac{b}{2}\right)$.

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$\Longrightarrow$ We use $L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$ ideals $\tilde{\mathfrak{a}} = \mathfrak{a}\prod \mathfrak{p}_i^{e_i}$ for small prime ideals $\mathfrak{p}_i$ and integers e_i to be sure to derive one $\tilde{\mathfrak{b}}$ that is $L_{|\Delta_{\mathbb{K}}|}(1)$-smooth.

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Cost: $L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$ for lattice reduction & smoothness tests

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2.3. The $\mathfrak{q}$-descent – The final step

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- The total runtime of the $\mathfrak{q}$-descent is $L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$.

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  Relation: principal ideal that splits on the factor base. Test ideals generated by $\boldsymbol{v} = \sum v_i(\zeta^i + \zeta^{-i})$ for $|v_i| \leq \log|\Delta_{\mathbb{K}}|$.

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- Relation collection: construction of a full-rank matrix M

  Relation: principal ideal that splits on the factor base. Test ideals generated by $\boldsymbol{v} = \sum v_i(\zeta^i + \zeta^{-i})$ for $|v_i| \leq \log|\Delta_{\mathbb{K}}|$.

  Norm below $L_{|\Delta_{\mathbb{K}}|}(1) \Longrightarrow L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$-smooth ideals in $L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$.

Input: Bunch of prime ideals of norm below $B = L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$

Index Calculus Method:

- Factor base: set of all prime ideals with norm below B
- Relation collection: construction of a full-rank matrix M

$$\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_{Q|\mathcal{B}|} \end{pmatrix} \begin{matrix} \rightarrow \\ \rightarrow \\ \vdots \\ \rightarrow \end{matrix} \begin{pmatrix} M_{1,1} & \cdots & M_{1,|\mathcal{B}|} \\ M_{2,1} & \cdots & M_{2,|\mathcal{B}|} \\ \vdots & & \vdots \\ M_{Q|\mathcal{B}|,1} & \cdots & M_{Q|\mathcal{B}|,|\mathcal{B}|} \end{pmatrix} \Longrightarrow \forall i, \langle \boldsymbol{v}_i \rangle = \prod_{j=1}^{|\mathcal{B}|} \mathfrak{p}_j^{M_{i,j}}$$

Input: Bunch of prime ideals of norm below $B = L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$

Index Calculus Method:

- Factor base: set of all prime ideals with norm below B
- Relation collection: construction of a full-rank matrix M
- A N-dimensional vector Y including all the valuations of the smooth ideals in the $\mathfrak{p}_i$

3. Solution for smooth ideals

Input: Bunch of prime ideals of norm below $B = L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$

Index Calculus Method:

- Factor base: set of all prime ideals with norm below B
- Relation collection: construction of a full-rank matrix M
- A N-dimensional vector Y including all the valuations of the smooth ideals in the $\mathfrak{p}_i$
- A solution X of $MX = Y$ provides a generator of the product of the $L_{|\Delta_{\mathbb{K}}|}\left(\frac{1}{2}\right)$-smooth ideals

Implementation results

`PARI-GP` and `fplll` for BKZ-reductions — Intel(R) Xeon(R) CPU E3-1275 v3 @ 3.50GHz with 32GB of memory

Dimension of the field: $N = 2^8 = 256$.

Implementation results

`PARI-GP` and `fplll` for BKZ-reductions — Intel(R) Xeon(R) CPU E3-1275 v3 @ 3.50GHz with 32GB of memory

Dimension of the field: $N = 2^8 = 256$.

- Gentry-Szydlo: 20h and 24GB memory

Implementation results

`PARI-GP` and `fplll` for BKZ-reductions — Intel(R) Xeon(R) CPU E3-1275 v3 @ 3.50GHz with 32GB of memory

Dimension of the field: $N = 2^8 = 256$.

- Gentry-Szydlo: 20h and 24GB memory
- BKZ-reduction: between 10 min and 4h
  (Descent reduced to only one step)

Implementation results

`PARI-GP` and `fplll` for BKZ-reductions — Intel(R) Xeon(R) CPU E3-1275 v3 @ 3.50GHz with 32GB of memory

Dimension of the field: $N = 2^8 = 256$.

- Gentry-Szydlo: 20h and 24GB memory
- BKZ-reduction: between 10 min and 4h
  (Descent reduced to only one step)

We recover $\boldsymbol{g} \cdot \zeta^i$ — and so the secret key $\boldsymbol{g}$ — in less than a day.

Thanks

Thank you