# Efficient compression of SIDH public keys

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# Supersingular-isogeny Diffie-Hellman

- Post-quantum secure (ephemeral) key exchange [JF11]
- Based on hardness of finding large-degree isogenies
- Small keys ($\approx 564$ bytes public)
- Relatively slow compared to other PQ proposals
- Key compression ($\approx 385$ bytes), at very high cost [Aza+16]

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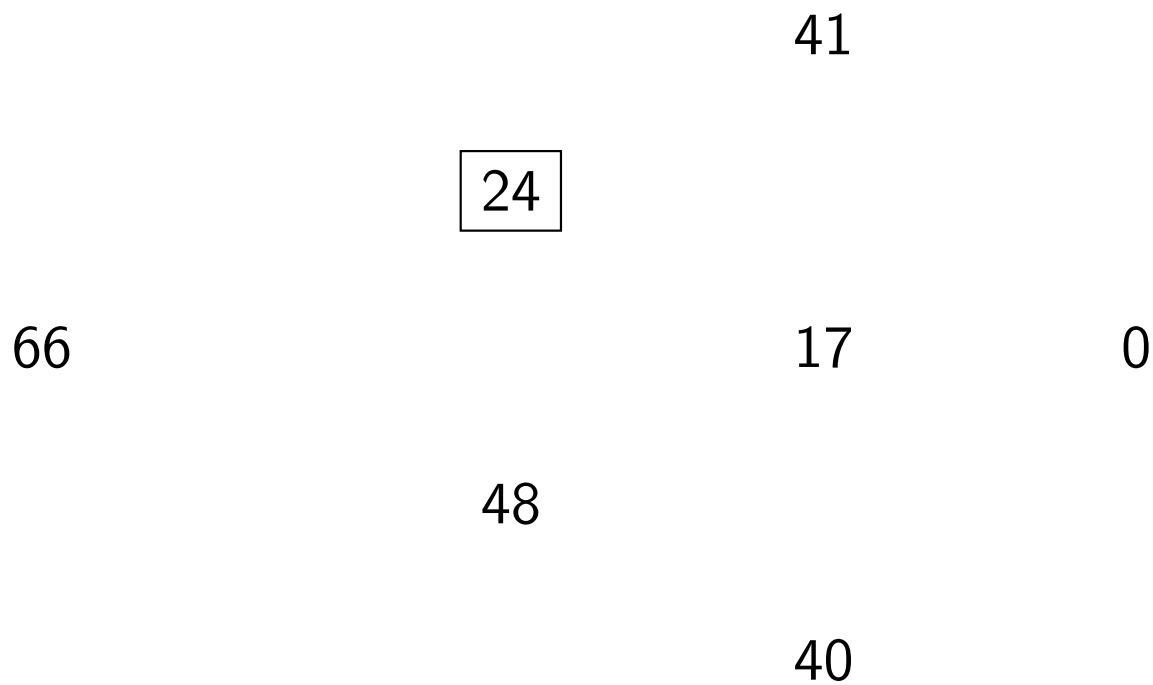
- Key compression ($\approx$ 385 bytes), at very high cost [Aza+16]

## This talk

- Key size reduced by 12.5% ($\approx$ 330 bytes)
- Compression up to 66$\times$ faster
- Decompression up to 15$\times$ faster

# Isogeny graphs

$$p = 2^3 \cdot 3^2 - 1, \quad E/\mathbb{F}_{p^2} : y^2 = x^3 + x, \quad j(E) = 24, \quad \ell = 2$$



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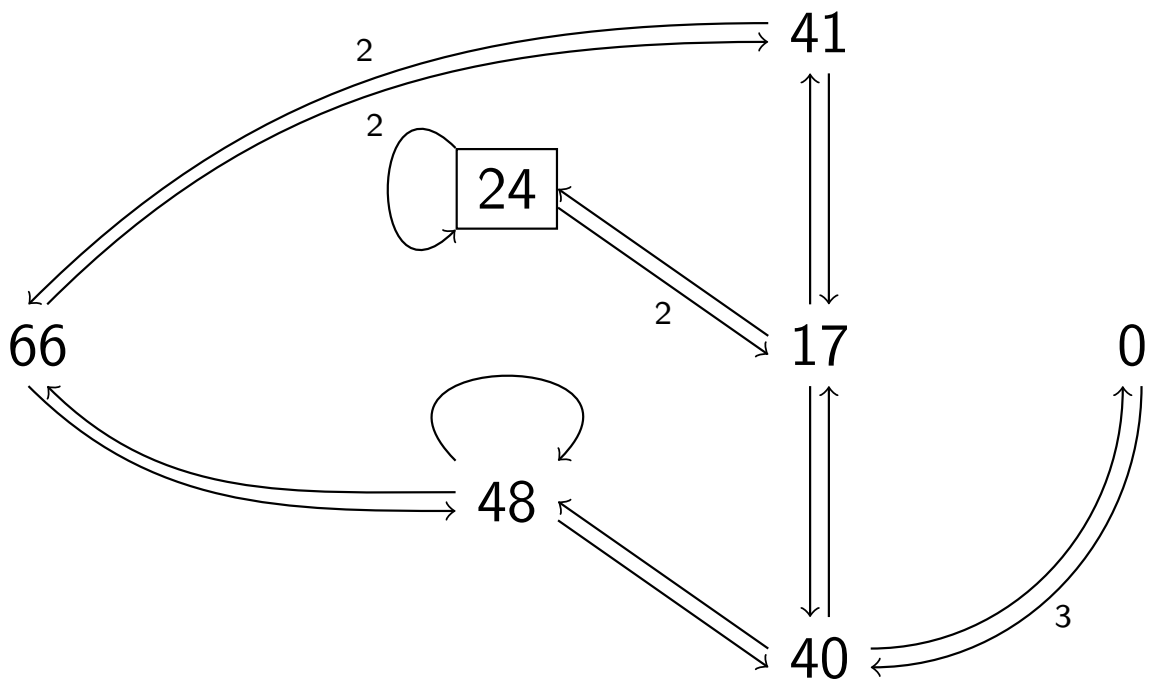

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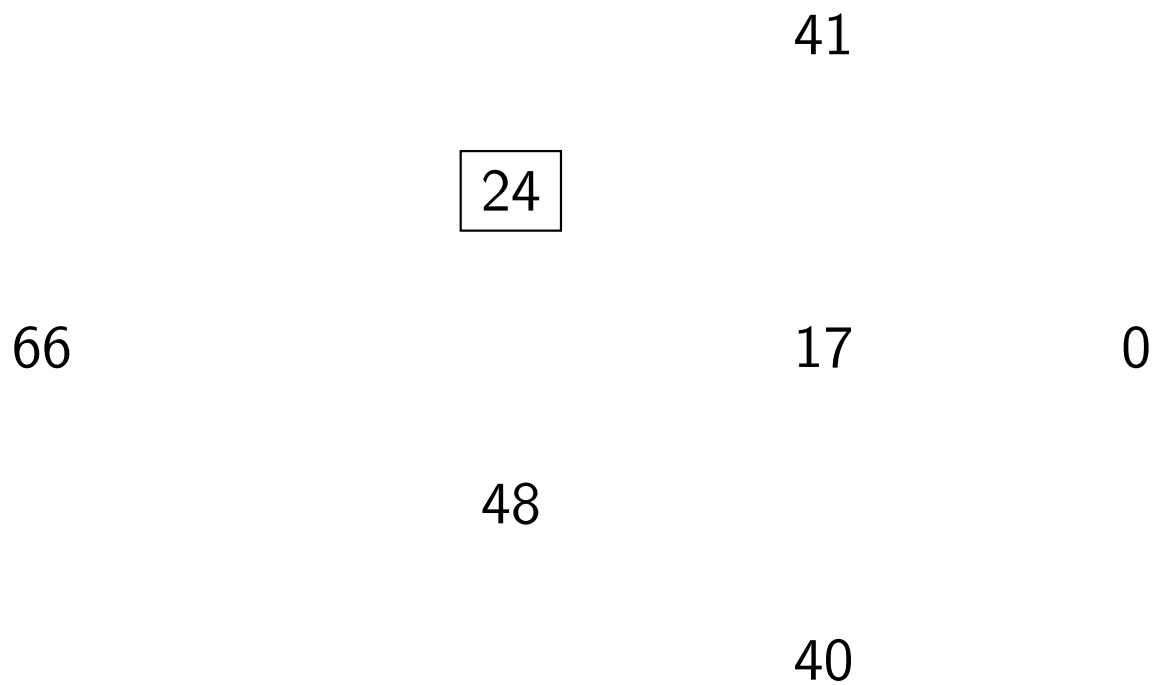
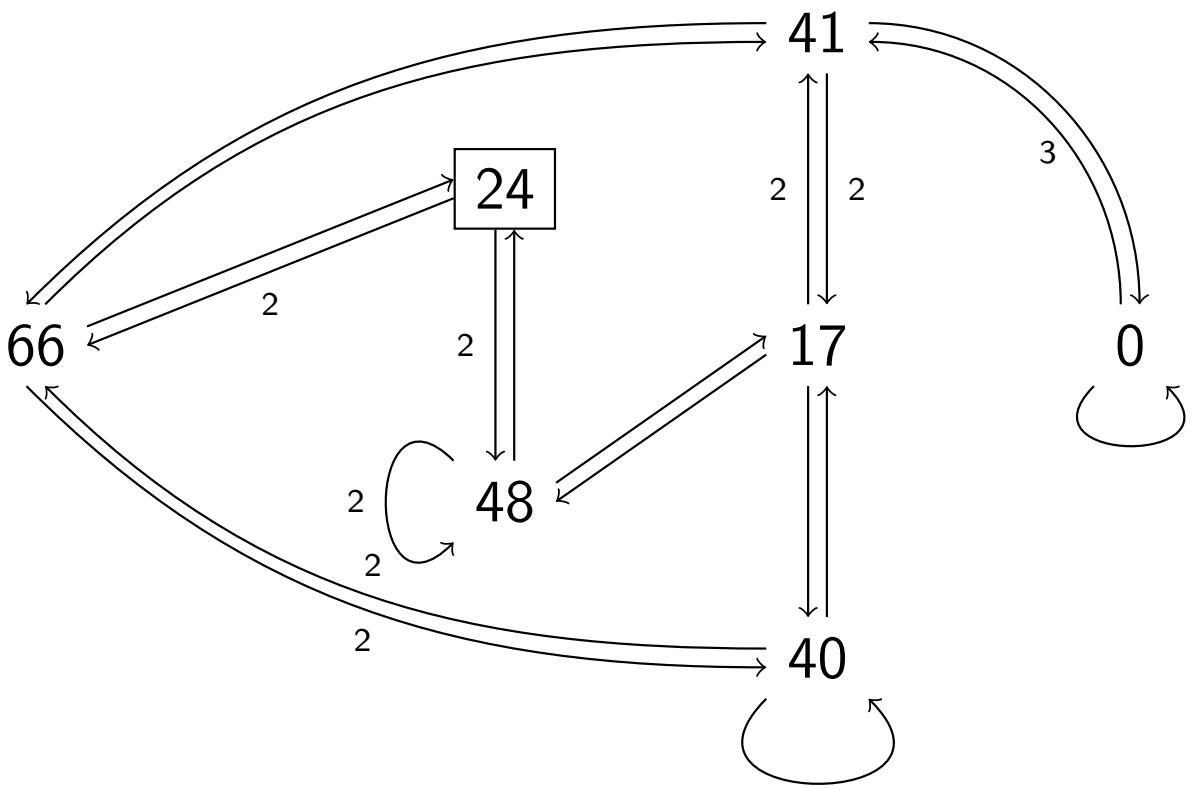


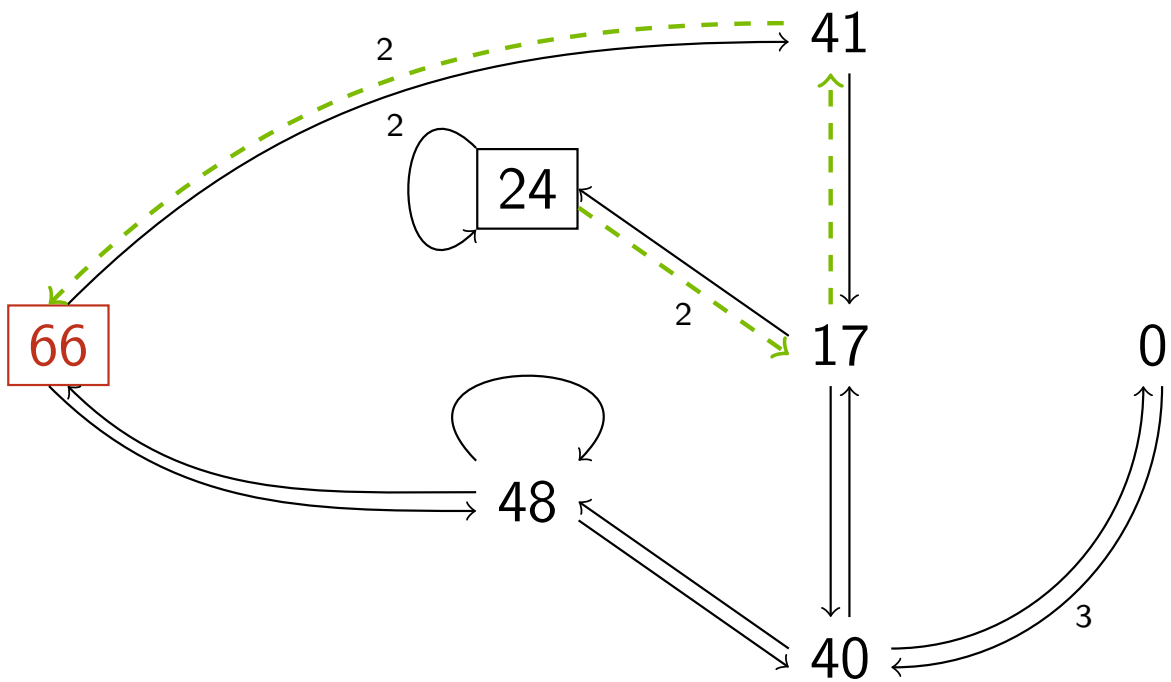
# Isogeny graphs

$$p = 2^3 \cdot 3^2 - 1, \quad E/\mathbb{F}_{p^2} : y^2 = x^3 + x, \quad j(E) = 24, \quad \ell = 3$$



# Key generation

= private party $A$, = private party $B$, = public keys



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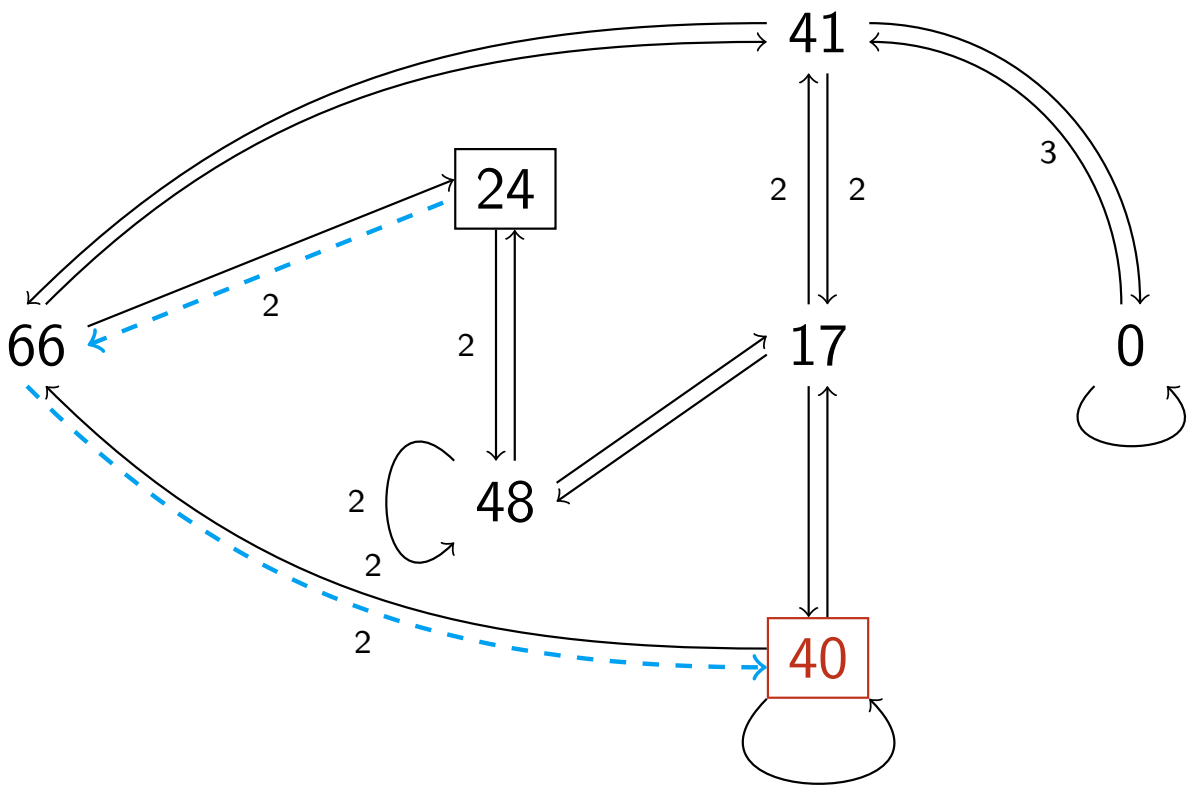

# Supersingular-isogeny Diffie-Hellman [JF11]

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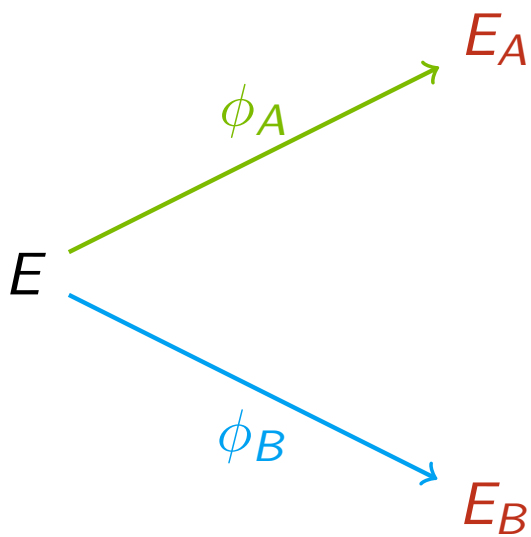

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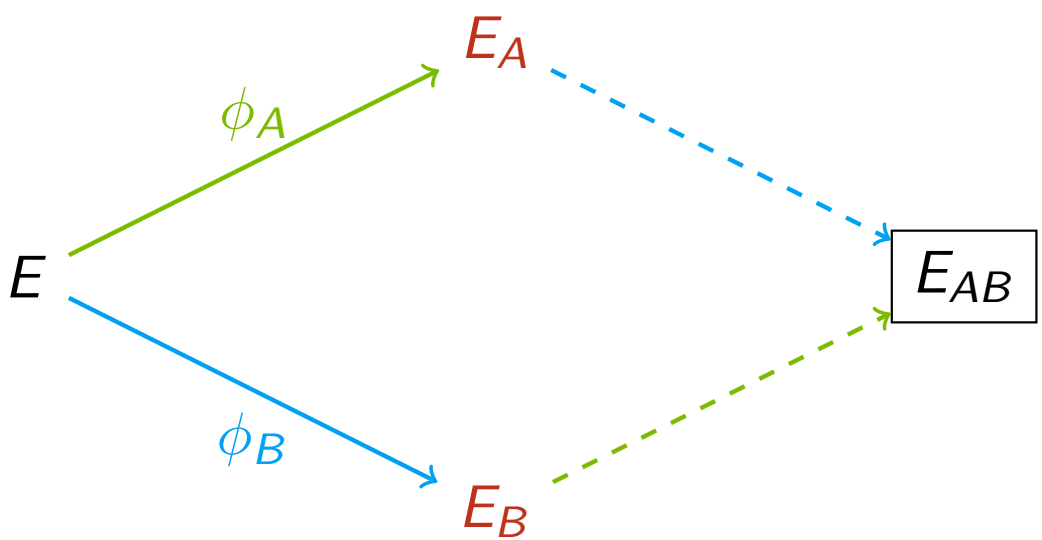

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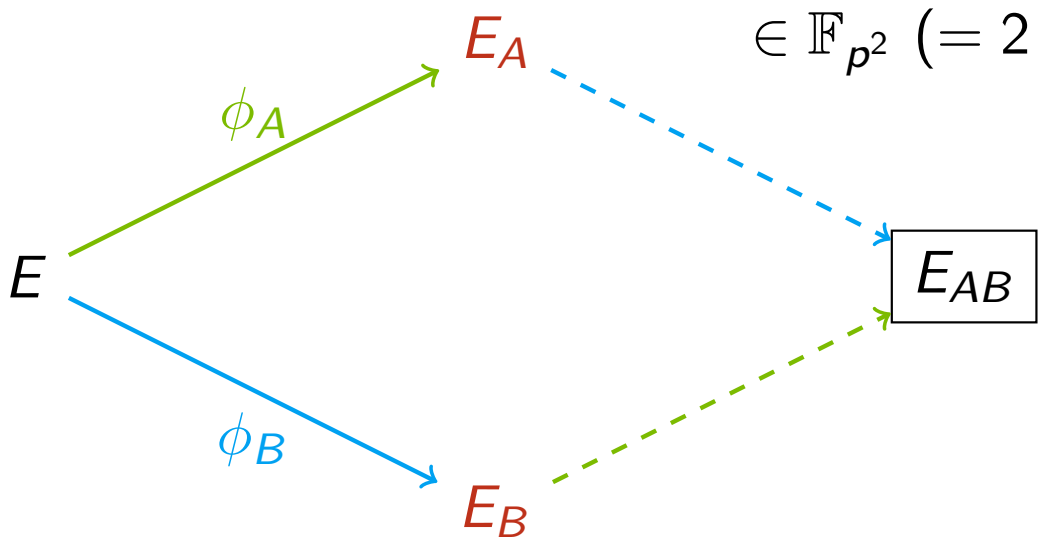
$E_A[\ell^e] = \langle P, Q \rangle \quad \in \mathbb{F}_{p^2}^2 \ (= 4 \log p \text{ bits})$

$\in \mathbb{F}_{p^2} \ (= 2 \log p \text{ bits})$



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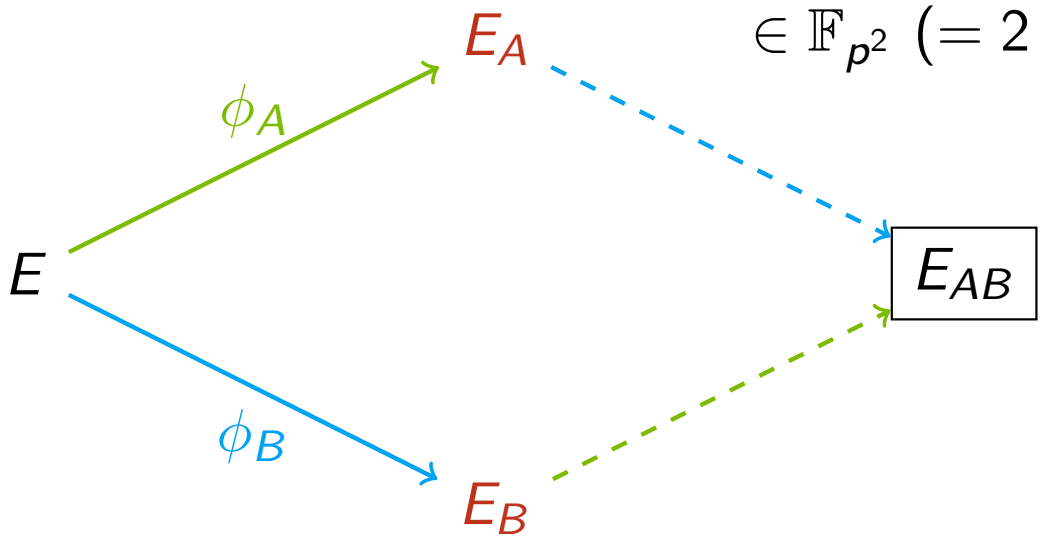

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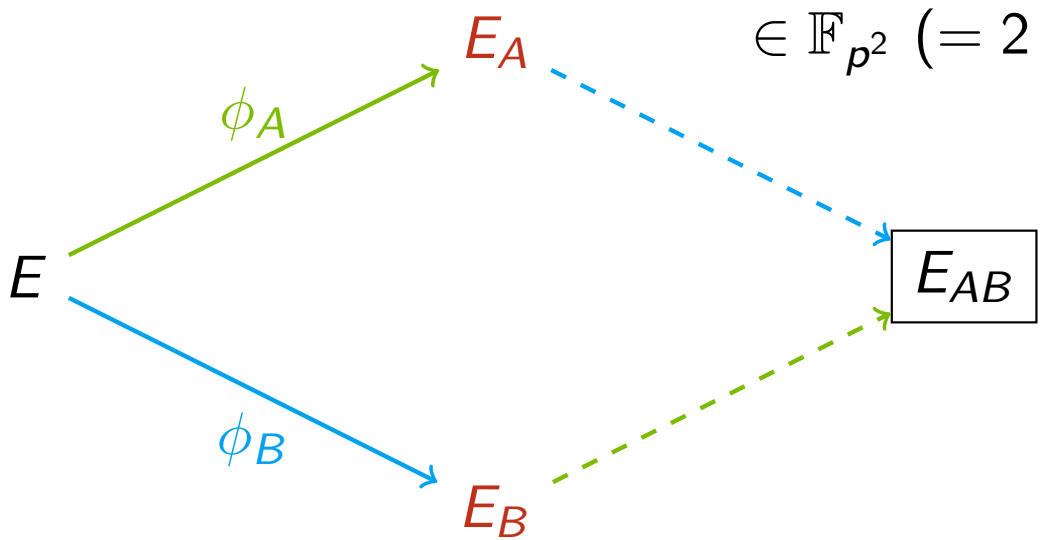
■ = private party $A$, ■ = private party $B$, ■ = public key

↗ = 2-graph walk, ↘ = 3-graph walk,

$$E_A[\ell^e] = \langle R, S \rangle$$

$(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}_{\ell^e}^4$ ($\approx 2 \log p$ bits)

$\in \mathbb{F}_{p^2}$ ($= 2 \log p$ bits)



# Public-key compression [Aza+16]

**Compression**

$$\langle P, Q\rangle \longrightarrow \begin{array}{c}\langle R, S\rangle \\ \langle \alpha R + \beta S, \gamma R + \delta S\rangle\end{array} \longrightarrow (\alpha, \beta, \gamma, \delta)$$

**Decompression**

$$(\alpha, \beta, \gamma, \delta) \longrightarrow \begin{array}{c}\langle R, S\rangle \\ (\alpha, \beta, \gamma, \delta)\end{array} \longrightarrow \langle P, Q\rangle$$

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**Compression**

$$\langle P, Q\rangle \longrightarrow \begin{array}{c}\langle R, S\rangle \\ \langle \alpha R + \beta S, \gamma R + \delta S\rangle\end{array} \xrightarrow{\text{Expensive}} (\alpha, \beta, \gamma, \delta)$$

**Decompression**

$$(\alpha, \beta, \gamma, \delta) \longrightarrow \begin{array}{c}\langle R, S\rangle \\ (\alpha, \beta, \gamma, \delta)\end{array} \longrightarrow \langle P, Q\rangle$$

# Public-key compression [Aza+16]

Significantly improve efficiency (up to 66×)

**Compression**

$$\langle P, Q\rangle \longrightarrow \begin{matrix}\langle R, S\rangle \\ \langle \alpha R + \beta S, \gamma R + \delta S\rangle\end{matrix} \longrightarrow (\alpha, \beta, \gamma, \delta)$$

**Decompression**

$$(\alpha, \beta, \gamma, \delta) \longrightarrow \begin{matrix}\langle R, S\rangle \\ (\alpha, \beta, \gamma, \delta)\end{matrix} \longrightarrow \langle P, Q\rangle$$

Significantly improve efficiency (up to 15×)

# Finding a canonical basis

Find $R, S$ such that $E[2^{372}] = \langle R, S \rangle$, where

$$\#E(\mathbb{F}_{p^2}) = \left(2^{372}3^{239}\right)^2.$$

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## Finding an element of order $2^{372}$

1. Deterministically pick $R \in E(\mathbb{F}_{p^2}) \setminus 2E(\mathbb{F}_{p^2})$

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## Finding an element of order $2^{372}$

1. Deterministically pick $R \in E(\mathbb{F}_{p^2}) \setminus 2E(\mathbb{F}_{p^2})$

For $E : y^2 = x(x-\gamma)(x-\delta)$,

$$R \in 2E(\mathbb{F}_{p^2}) \iff x_R, x_R - \delta, x_R - \gamma \text{ are squares}$$

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## Finding an element of order $2^{372}$

1. Deterministically pick a non-square $x_R \in \mathbb{F}_{p^2}$
2. If $x_R^3 + Ax_R^2 + x_R$ is not a square, goto 1

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1. Deterministically pick a non-square $x_R \in \mathbb{F}_{p^2}$
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3. Set $R \leftarrow (x_R, \sqrt{x_R^3 + Ax_R^2 + x_R})$

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4. Set $R \leftarrow [3^{239}]R$

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## Finding a canonical basis of $E[2^{372}]$

1. Pick $R \in E(\mathbb{F}_{p^2})$ of order $2^{372}$
2. Pick $S \in E(\mathbb{F}_{p^2})$ of order $2^{372}$
3. If $E[2^{372}] \neq \langle R, S \rangle$, goto 2.

# Transferring to $\mu_n$ via reduced Tate pairing

Transfer the discrete logs to $\mu_n$

$$e = e(R, S) \qquad e^{\beta} = e(R, P) \qquad e^{\delta} = e(R, Q)$$
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such that $P = \alpha R + \beta S$ and $Q = \gamma R + \delta S$

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---

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$f_0 \leftarrow f_{n,R}$ $f_3 \leftarrow f_{n,S}$ $f_4 \leftarrow f_{n,S}$

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$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$

**Optimized formulas for $f_{n,R}$ and $f_{n,S}$!**

# Efficient discrete logarithms (Pohlig-Hellman)

For $e_0, e_1, e_2, e_3, e_4 \in \mu_{\ell^e}$, compute $\alpha, \beta, \gamma, \delta$ such that

$$e_1 = e_0^{-\alpha}, \qquad e_2 = e_0^{\beta}, \qquad e_3 = e_0^{-\gamma}, \qquad e_4 = e_0^{\delta}$$

As $\mu_{\ell^e} \subset G_{p+1} \subset \mathbb{F}_{p^2}$, $\mathbf{I} \approx \mathbf{M}$, $\mathbf{S} \approx 2\mathbf{s}$, $\mathbf{C} \approx 2\mathbf{m} + 1\mathbf{s}$

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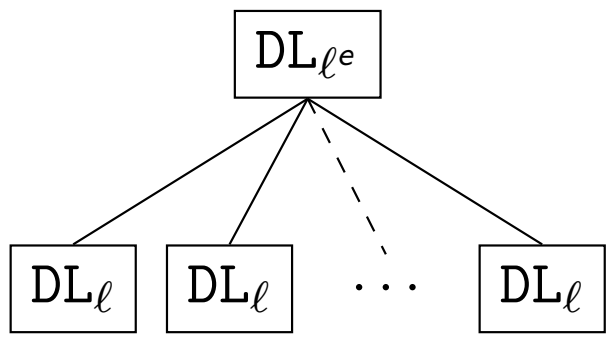
$\#G_1 = \ell^e$

$\#G_1 = \ell$



# Nested Pohlig-Hellman

$$\#G_1 = \ell^{e_1}$$

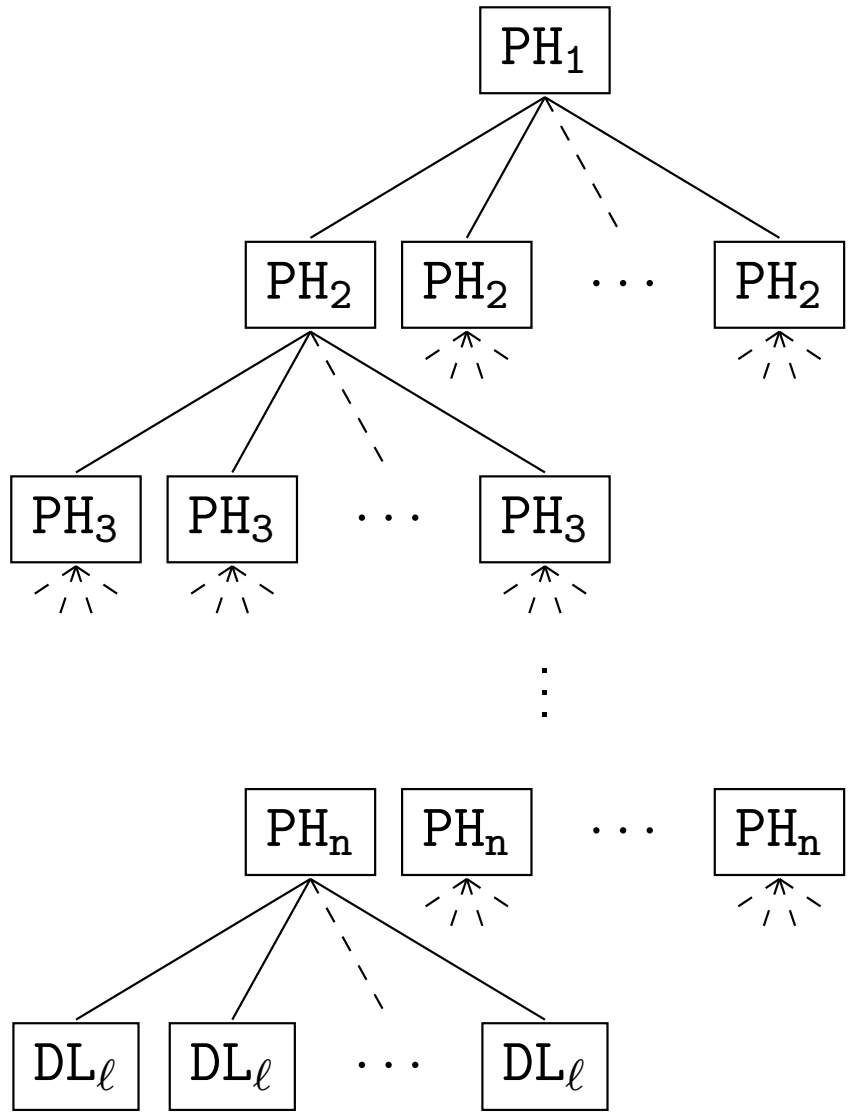
$$\#G_2 = \ell^{e_2}$$

$$\#G_3 = \ell^{e_3}$$

$$\vdots$$

$$\#G_n = \ell^{e_n}$$

$$\#G_{n+1} = \ell$$



# Comparison

| # | windows | | | | $\mathbb{F}_{p^2}$ | | table size |
|---|---|---|---|---|---|---|---|
| $n$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | **M** | **S** | $\mathbb{F}_{p^2}$ |
| 0 | – | – | – | – | 372 | 69 378 | 375 |
| 1 | 19 | – | – | – | 375 | 7 445 | 43 |
| 2 | 51 | 7 | – | – | 643 | 4 437 | 25 |
| 3 | 84 | 21 | 5 | – | 716 | 3 826 | 25 |
| 4 | 114 | 35 | 11 | 3 | 1 065 | 3 917 | 27 |

Options for different time-memory trade-offs [Sut11]

# Signature size reduction

- The quadruple $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}_{\ell^e}^4$ determines

$$P = \alpha R + \beta S, \quad Q = \gamma R + \delta S.$$

  These determine $\langle P + \lambda Q \rangle$, for some $\lambda \in \mathbb{Z}_{\ell^e}^*$
- Thus we only need $P, Q$ up to scalar, and compress to

$$[\alpha : \beta : \gamma : \delta] .$$

  As $P, Q$ form a basis of $E[\ell^e]$, either $\alpha$ or $\beta$ is invertible
- Normalizing, we represent it in $\mathbb{Z}_{\ell^e}^3 \times \mathbb{Z}_2$

# Benchmarks (for $\ell = 2$)

| | **This work** | [Aza+16] | Speed-up |
|---|---|---|---|
| Key size (bytes) | 328 | 385 | – |
| SIDH (cc $\times$ $10^6$) | 80 | – | – |
| Compression (cc $\times$ $10^6$) | 109 | 6 081 | 56$\times$ |
| Decompression (cc $\times$ $10^6$) | 42 | 539 | 13$\times$ |
| Full no comp. (cc $\times$ $10^6$) | 192 | 535 | 2.8$\times$ |
| Full comp. (cc $\times$ $10^6$) | 469 | 15 395 | 31$\times$ |

Software available at

https://github.com/Microsoft/PQCrypto-SIDH

# Thanks!

Questions?

# References I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